A NONSTATIONARY PROBLEM OF COUNTERCURRENT MIXTURE SEPARATION IN A PLANT WITH DISTRIBUTED PARAMETERS

G. G. Filippov and M. P. Nekrasov

Inzhenerno-Fizicheskii Zhurnal, Vol. 13, No. 1, pp. 86–89, 1967

UDC 536.70

A solution that can be used in designing an absorption and extraction plant has been found for system of equations (1) describing the processes of countercurrent separation of binary mixtures for initial and boundary conditions (2).

Such separation processes as absorption, extraction, distillation, and ion exchange are often carried out in apparatus whose parameters may be assumed to be uniformly distributed along the length. The material balance of an element of length is usually constructed for each component of the separated mixture in each phase. Thus, for a binary mixture under steady-state hydrodynamic conditions, neglecting longitudinal mixing and transverse nonuniformity, we obtain

$$L\frac{\partial x}{\partial z}+h_L\frac{\partial x}{\partial t}-R(x;\ y)=0,$$
$$V\frac{\partial y}{\partial z}-h_V\frac{\partial y}{\partial t}-R(x;\ y)=0. \qquad (1)$$

The first equation of system (1) is the material balance of the required component in what we will call the heavy phase, the second the balance for the light phase.

It is possible to distinguish several types of boundary and initial conditions for system (1). The conditions typical of fractionating columns have been most carefully studied. Absorption and extraction are characterized by the following boundary conditions at $t \ge 0$:

$$y(0;\ 0)=y_0,$$
$$x(Z;\ 0)=x_0. \qquad (2)$$

System (1) is usually solved by reducing it to a single second-order partial differential equation, to which a Laplace transformation is then applied, the solutions being obtained in the form of an infinite series [1, 2]. Another approach, suggested by Bowmen and Briant [3], consists in going over from z and t to new independent variables of the form $\xi(z;t)$ and $\eta(z;t)$ in the course of reducing system (1) to a single equation. As Thomas has shown [4] with reference to a special case ($L = 0$), it is then possible to obtain a solution in finite form expressed in terms of tabulated functions. We will employ the method developed in [4] to solve system (1) for boundary conditions (2).

We go over from the variables z and t to the new variables ξ and η, defined as follows:

$$\xi=-h_Lz+Lt,$$
$$\eta=h_Vz+Vt. \qquad (3)$$

Substitution into (1) gives

$$\frac{\partial x}{\partial \eta}=\frac{R(x;\ y)}{Vh_L+Lh_V}=\frac{\partial y}{\partial \xi}. \qquad (4)$$

Equation (4) is a necessary and sufficient condition for existence of the function F given by the equation

$$dF=x\,d\xi-y\,d\eta, \qquad (5)$$

from which it follows that

$$x=\frac{\partial F}{\partial \xi};\quad y=-\frac{\partial F}{\partial \eta}. \qquad (6)$$

From (4) and (6) we obtain

$$\frac{\partial^2 F}{\partial \xi\,\partial \eta}=\frac{R(x;\ y)}{Vh_L+Lh_V}. \qquad (7)$$

We must now determine the form of the function R(x; y). In accordance with the theory of mass transfer for the region of low concentrations

$$R=K(\alpha y-x),$$

where K is the volume mass transfer coefficient, mole/($m^3\cdot$ sec); $1/\alpha$ is the distribution coefficient. Introducing the abbreviations

$$A=\frac{K}{Vh_L+Lh_V};\quad B=\frac{\alpha K}{Vh_L+Lh_V},$$

we rewrite (7) in the form:

$$\frac{\partial^2 F}{\partial \xi\,\partial \eta}+A\frac{\partial F}{\partial \xi}+B\frac{\partial F}{\partial \eta}=0. \qquad (8)$$

In accordance with [4] we make the substitution

$$F(\xi;\ \eta)=\exp[-(B\xi+A\eta)]\,\psi(\xi,\ \eta); \qquad (9)$$

then (8) assumes the form

$$\frac{\partial^2 \psi}{\partial \xi\,\partial \eta}=AB\psi. \qquad (10)$$

From (6) and (9) we obtain equations for expressing the concentrations in terms of ψ:

$$y=\exp[-(B\xi+A\eta)]\left(\psi A-\frac{\partial \psi}{\partial \eta}\right),$$
$$x=\exp[-(B\xi+A\eta)]\left(\psi B-\frac{\partial \psi}{\partial \xi}\right). \qquad (11)$$

Boundary conditions of the type in (2) go over into the following:

$$\psi(\xi;\ 0) = (1 + x_0\xi)\exp(B\xi),$$
$$\psi(0;\ \eta) = (1 - y_0\eta)\exp(A\eta). \qquad (12)$$

We apply a Laplace transformation to Eq. (10) with boundary conditions (12):

$$\psi^*(\xi;\ p) = p\int_0^\infty \psi(\xi;\ \eta)\exp(-p\eta)\,d\eta,$$

whence

$$\psi^*(\xi;\ p) = \exp(B\xi)\times$$
$$\times\left[\frac{p}{p-A} - \frac{x_0A}{B}\,\frac{p}{(p-A)^2} + \frac{x_0\xi p}{p-A}\right] +$$
$$+\exp(AB\xi/p)\left[\frac{Ax_0 - y_0B}{B}\,\frac{p}{(p-A)^2}\right].$$

The inverse transform of the function ψ is given by the following complex integral:

$$\psi(\xi;\ \eta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\psi^*(\xi;\ p)}{p}\exp(p\eta)\,dp,$$

which may be represented thus:

$$\psi(\xi;\ \eta) = \exp(B\xi + A\eta)\left(1 + x_0\xi - \frac{x_0A}{B}\,\eta\right) +$$
$$+\frac{x_0A - y_0B}{AB}\left\{\eta\,\frac{\partial}{\partial\eta} + \xi\,\frac{\partial}{\partial\xi}\right\}\varphi(A\eta;\ B\xi)\Big). \qquad (13)$$

From (11) and (13), using the properties of $\varphi(\eta;\ \xi)$ [4]:

$$\frac{\partial\varphi(\eta;\ \xi)}{\partial\eta} = \varphi(\eta;\ \xi) + I_0(2\sqrt{\eta\xi}),$$
$$\frac{\partial\varphi(\eta;\ \xi)}{\partial\xi} = \varphi(\eta;\ \xi) - \frac{\partial}{\partial\xi}\,I_0(2\sqrt{\eta\xi}),$$

which follow from its definition:

$$\varphi(\eta;\ \xi) = \exp(\eta)\int_0^\eta \exp(-t)\,I_0(2\sqrt{\xi t})\,dt,$$

we obtain the following expressions:

$$y = \frac{Ax_0}{B} + \exp\left[-(B\xi + A\eta)\right]\{y_0\left[I_0(2\sqrt{AB\xi\eta}) +\right.$$
$$\left.+\varphi(A\eta;\ B\xi)\right] -$$
$$-\frac{Ax_0}{B}\left[I_0(2\sqrt{AB\xi\eta}) + \varphi(A\eta;\ B\xi)\right]\}, \qquad (14)$$

$$x = \frac{By_0}{A} + \exp\left[-(B\xi + A\eta)\right]\times$$
$$\times\left[x_0\,\varphi(A\eta;\ B\xi) - \frac{By_0}{A}\,\varphi(A\eta;\ B\xi)\right]. \qquad (15)$$

Here, $I_0(2(\xi\eta)^{1/2})$ is a Bessel function of zero order and purely imaginary argument. The integral for $\varphi(\eta;\ \xi)$ can be evaluated for almost all values of η and ξ, except very small ones, using the following asymptotic expansion [4]:

$$\varphi(\eta;\ \xi) = \frac{1}{2}\left[1 - \mathrm{H}(\sqrt{\xi} - \sqrt{\eta})\right]\times$$
$$\times\exp(\eta + \xi) - \frac{r}{r+1}\,I_0(2\sqrt{\eta\xi}),$$

in which $r = (\eta/\xi)^{1/4}$, and $\mathrm{H}((\xi)^{1/2} - (\eta)^{1/2})$ is the error function.

Equations (14) and (15) also give a solution of the nonstationary problem for Eq. (10) and boundary conditions (12) and consequently for system (1).

L is the heavy-phase flow (mole/m^2 · sec); h_L is the heavy-phase holdup (mole/m^3); V is the light-phase flow (mole/m^2 · sec); h_V is the light-phase holdup (mole/m^3); R(x; y) is the amount of separated component passing from heavy into light phase (mole/m^3 · · sec); x is the concentration of separated component in heavy phase and y in light phase; t is time (sec); z is the distance along length of apparatus (m); Z is the total length of apparatus (m).

REFERENCES

1. K. Cohen, The Theory of Isotope Separation, McGraw-Hill, New York, 1951.

2. A. M. Rozen, Theory of Isotope Separation in Cascades [in Russian], Atomizdat, 1960.

3. J. R. Bowmen and R. C. Briant, Ind. Engng. Chem., 39, 745, 1947.

4. H. Thomas, collection: Ion Exchange [Russian translation], IL, p. 34, 1951.

25 October 1966

Karpov Physicochemical Institute, Moscow